Average genus and average signature of 2-bridge knots

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Knot Theory of Random Models, Banff IRS, April 5th, 2024

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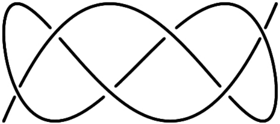

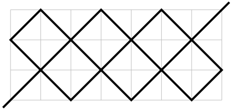

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Motivating question

Suppose we have a random knot model or experimental data.

How do we know whether this model or data accurately depicts

the behaviors of large knots?

We need some sort of baseline for comparison.

Main Results

For a given c, consider all 2-bridge knots with c crossings.

Part I Main Theorem [C.-Lowrance], [Suzuki-Tran]:

The average Seifert genus is asymptotically linear:

$$\overline{g}(c) = \frac{c}{4} + \frac{1}{12} + \varepsilon(c),$$

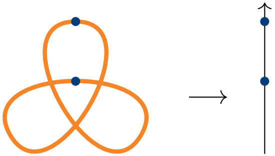
where $\varepsilon(c) \to 0$ as $c \to \infty$.

Part III Main Theorem [C.-Lowrance-Raanes]:

The average absolute value of the signature $\overline{\sigma}(c)$ satisfies:

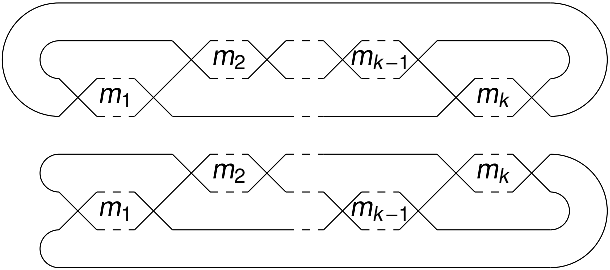
$$\lim_{c\to\infty}\left(\overline{\sigma}(c) - \sqrt{\frac{2c}{\pi}}\right) = 0.$$

The ***bridge number*** is the minimum number of local max/min of any height function on any diagram of the knot.



Today we restrict our attention to knots with bridge number 2.

A ***2-bridge knot*** has one of the following forms (by parity):



with the ith ***twist region*** having m_i crossings.

All 2-bridge knots are ***alternating knots***.

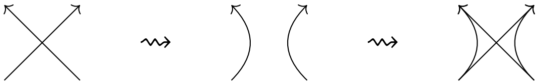
Our 2-bridge knots will have all $m_{2i+1} > 0$ with each ╳ written σ_1 and all $m_{2i} < 0$ with each ╳ written σ_2^{-1}.

The ***Seifert genus*** $g(K)$ of a knot K is the minimum genus
of an orientable surface whose boundary is K
taken over all such surfaces.

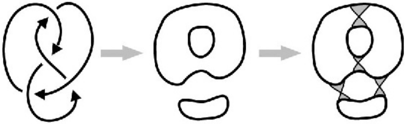
Theorem [Frankl-Pontyagin 1930], [Seifert 1934]:

Every knot has some such surface, a ***Seifert surface***.

[Seifert]'s algorithm:



Cap off each circle with a disk. Attach twisted bands at crossings.



From *Visualization of Seifert surfaces* **[Wijk-Cohen 2006]**

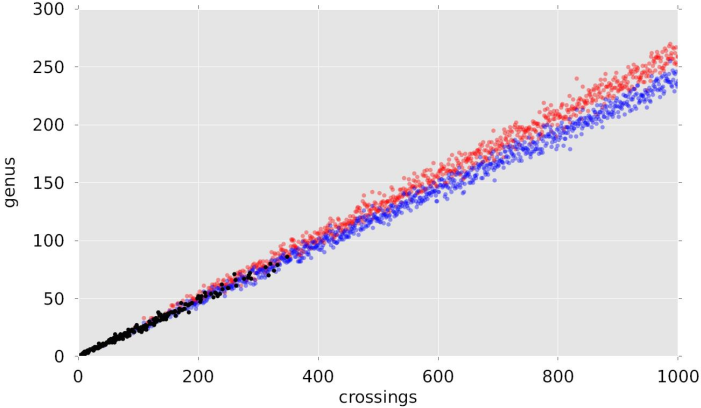
Let s be the number of ***Seifert circles*** of a knot diagram.

***Theorem* [Murasugi 1958], [Crowell 1959]:**

The Seifert genus of an alternating knot K with c crossings is the genus obtained from applying Seifert's algorithm to an alternating diagram D with

$$g(K) = g(D) = \frac{c+1-s}{2}.$$

[Dunfield et al] give experimental data that suggests that



the ***genus*** grows linearly w.r.t. ***crossing number***.

I. Average Seifert genus

Motivating ideas: On expected genus

Theorem [Brooks-Makover 2004, Gamburd-Makover 2002]:
for a random Riemann surface.

Theorem [Linial-Nowik 2011]: for a random chord diagram.

Part I Main Theorem [C.-Lowrance], [Suzuki-Tran]:

The average Seifert genus $\overline{g}(c)$ over all 2-bridge knots with c crossings is asymptotically linear w.r.t. the crossing number c:

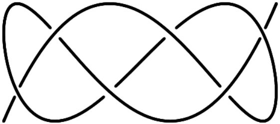
$$\overline{g}(c) = \frac{c}{4} + \frac{1}{12} + \varepsilon(c),$$

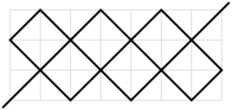
where $\varepsilon(c) \to 0$ as $c \to \infty$.

I. Average Seifert genus

***Background diagrammatics** [Koseleff-Pecker 2010s]:*

Chebyshev parametrizations and billiard table diagrams.





***Theorem** [C.-Krishnan 2015], [C.–Even-Zohar–Krishnan 2018]:*

A random model for 2-bridge knots.

Number the crossings from left to right.

A crossing oriented horizontally is denoted H.

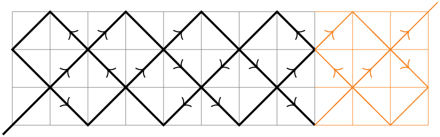
A crossing oriented vertically is denoted V.

Proposition [C. 2021, Property 7.1 on writhe]:

In a billiard table diagram with $a = 3$:

for $n = 3m$ crossings, they are oriented $(VHV)^m$, and

for $n = 3m + 1$ crossings, they are oriented $H(VVH)^m$.



Theorem [C.-Krishnan 2015], [C.–Even-Zohar–Krishnan 2018]:

Obtain “reduced” words, avoiding over-counting.

Consider the ***partially-double-counted set*** $T(c)$
of 2-bridge knots with c ***crossings***
only counts palindromic knots once.

Theorem [C. 2023]:

A lower bound for average Seifert genus over $T(c)$.

Theorem [C.-Lowrance]:

The size $t(c)$ of the set $T(c)$ satisfies $t(c) = t(c-1) + 2t(c-2)$,
giving the Jacobsthal numbers.

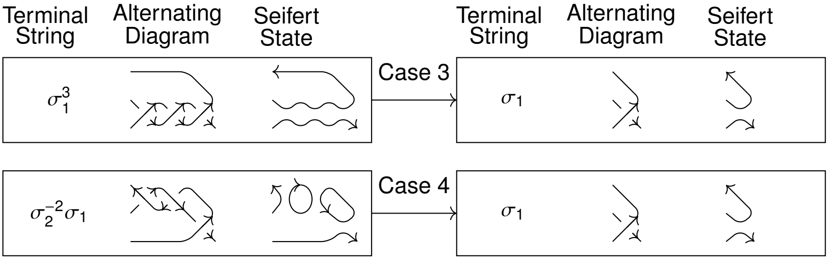
Main proof ingredient [C.-Lowrance]:

Bijection between $T(c)$ and $T(c-1) \sqcup T(c-2) \sqcup T(c-2)$
by considering the last three crossings (here for odd c):

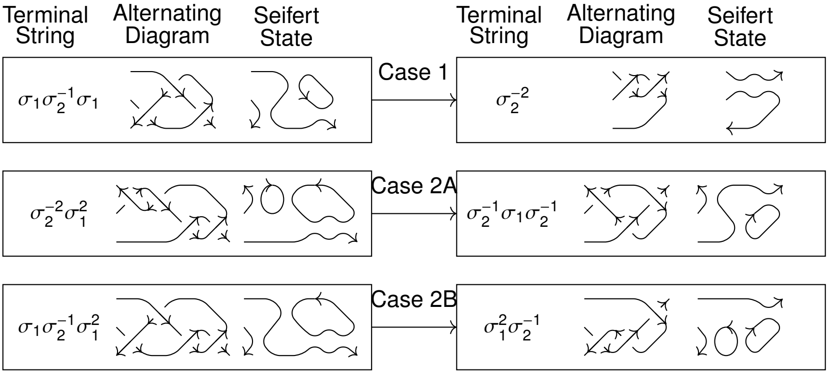
Case	Word in $T(c)$	$\mapsto$	Word	in Set
(1)	$w\sigma_1\sigma_2^{-1}\sigma_1$	$\mapsto$	$w\sigma_2^{-1}\sigma_2^{-1}$	$T(c-1)$,
(2)	$w\sigma_2^{-1}\sigma_1\sigma_1$	$\mapsto$	$w\sigma_1\sigma_2^{-1}$	$T(c-1)$,
(3)	$w\sigma_1\sigma_1\sigma_1$	$\mapsto$	$w\sigma_1$	$T(c-2)$, and
(4)	$w\sigma_2^{-1}\sigma_2^{-1}\sigma_1$	$\mapsto$	$w\sigma_1$	$T(c-2)$.

The first two cases partition $T(c-1)$,
and the last two cases give two copies of $T(c-2)$.

I. Average Seifert genus



I. Average Seifert genus



Theorem [C.-Lowrance]:

Let $s(c)$ be the total number of Seifert circles for knots in $T(c)$. Then $s(c)$ satisfies the recursion

$$s(c) = s(c-1) + 2s(c-2) + 3t(c-2).$$

We repeat the process for the palindromic knots, add them in, and divide by two.

Part I Main Theorem [C.-Lowrance], [Suzuki-Tran]:

The average Seifert genus $\overline{g}(c)$ over all 2-bridge knots with c crossings is asymptotically linear w.r.t. the crossing number c:

$$\overline{g}(c) = \frac{c}{4} + \frac{1}{12} + \varepsilon(c),$$

where $\varepsilon(c) \to 0$ as $c \to \infty$.

II. Distribution of genera approaches normal

Motivating ideas: On genus distribution of surfaces

Theorem [Chmutov-Pitel 2013, 2016]:

Randomly glue the sides of an *n*-gon: asymptotically normal.

Glue the sides of multiple polygonal disks: asymptotically normal.

Theorem [Even-Zohar–Farber 2021]:

Glue *some* of the sides together (surface with boundary):
asymptotically a bivariate normal distribution.

Theorem [Shrestha 2022]:

Square-tiled surfaces: satisfy a local central limit theorem.

II. Distribution of genera approaches normal

Part II Main Theorem
[C.-DiNardo-Lowrance-Raanes-Rivera-Steindl-Wanebo]:

The distribution of genera of 2-bridge knots with crossing number c approaches a normal distribution as c approaches ∞.

Questions: What does this say about

genus? 2-bridge knots? all knots? the normal distribution?

Thm [C.-DiNardo-Lowrance-Raanes-Rivera-Steindl-Wanebo]:

The median and mode are $\lfloor \frac{c+2}{4} \rfloor$.

The variance approaches $\frac{c}{16} - \frac{177}{44}$ as c approaches ∞.

II. Distribution of genera approaches normal

***Thm** [C.-DiNardo-Lowrance-Raanes-Rivera-Steindl-Wanebo]:*

The # of 2-bridge knots with crossing number c and genus g is

$$\bar{t}(c,g) = \frac{1}{2}\left((-1)^{f_1(c,g)} \sum_{n=0}^{f_1(c,g)} (-1)^n \binom{n+g-1}{n} + (-1)^{f_2(c,g)} \sum_{n=0}^{f_2(c,g)} (-1)^n \binom{n+2g-1}{n} \right),$$

where $f_1(c,g) = \lfloor \frac{c+1}{2} \rfloor - g - 1$ and $f_2(c,g) = c - 2g - 1$,

for $1 \leq g \leq \lfloor \frac{c-1}{2} \rfloor$ and 0 otherwise.

Thm [C.-DiNardo-Lowrance-Raanes-Rivera-Steindl-Wanebo]:

The # of 2-bridge knots with crossing number c and genus g is

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where $f_1(c,g) = \lfloor \frac{c+1}{2} \rfloor - g - 1$ and $f_2(c,g) = c - 2g - 1$,

for $1 \leq g \leq \lfloor \frac{c-1}{2} \rfloor$ and 0 otherwise.

III. Average (absolute value of) signature

From a knot diagram D we obtain a Seifert surface S as before.

We create the ***Seifert matrix*** M whose entries are
linking numbers of basis elements in $H_1(S)$.

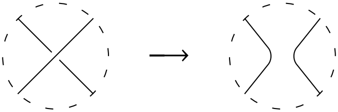
Symmetrize this matrix $M + M^T$.

The ***signature of the matrix*** $M + M^T$ is the difference between
the numbers of positive and negative eigenvalues.

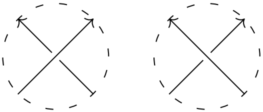
The ***signature of a knot*** K is
the signature of any symmetrized Seifert matrix.

III. Average (absolute value of) signature

Let $s_A(D)$ be the number of circles in the ***all-A-smoothing*** of D.



Let $n_+(D)$ be the number of ***positive crossings*** of D.

positive  negative

Theorem [Traczyk 2004]:

The signature of an alternating knot K with c crossings as obtained from an alternating diagram D is

$$\sigma(K) = s_A(D) - n_+(D) - 1.$$

III. Average (absolute value of) signature

Part III Main Theorem [C.-Lowrance-Raanes]:

The average absolute value of the signature $\overline{\sigma}(c)$ satisfies

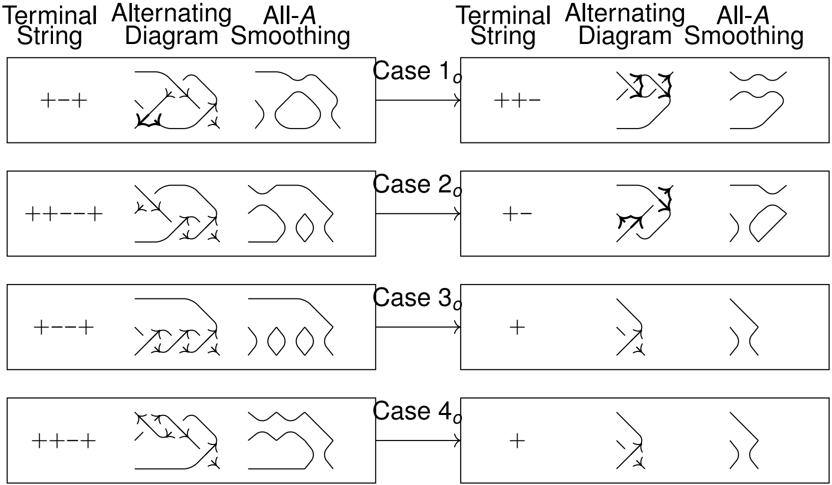
$$\lim_{c\to\infty}\left(\overline{\sigma}(c) - \sqrt{\frac{2c}{\pi}}\right) = 0.$$

Theorem [C.-Lowrance-Raanes]:

The number $s(c,\sigma)$ of words in $T(c)$ corresponding to a knot with signature σ satisfies the recurrence relation

$$s(c,\sigma) = s(c-1,\sigma+(-1)^c 2) + s(c-2,\sigma+(-1)^c 2) + s(c-2,\sigma).$$

III. Average (absolute value of) signature



$c\backslash\sigma$	-10	-8	-6	-4	-2	0	2	4	6	8	10	12
3							1					
4						1						
5							2	1				
6					1	3	1					
7						1	5	4	1			
8				1	5	9	5	1				
9					1	6	15	14	6	1		
10			1	7	20	29	20	7	1			
11				1	8	27	50	49	27	8	1	
12		1	9	35	76	99	76	35	9	1		
13			1	10	44	111	176	175	111	44	10	1
14	1	11	54	155	286	351	286	155	54	11	1	

The number $s(c, \sigma)$ of words in $T(c)$ with signature σ.

The $\#$ of 2-bridge knots with crossing number c and signature σ in $T(c)$ we call $s(c,\sigma)$.

Theorem [C.-Lowrance-Raanes]:

The sum of these entries in an odd and successive even row gives:

$$s(2m+1,\sigma)+s(2m+2,\sigma)=\binom{2m-1}{k}.$$

Frustration:

We don't have a simple combinatorial proof.

Other related work

[Clark-Frank-Lowrance, Suzuki-Tran]
On the braid indices of 2-bridge knots.

With Kindred, Lowrance, Van Cott, and Shanahan.

Similar work for other knot invariants.

Some references to my work

- *[C.–Krishnan 2015]*,
  Random knots using Chebyshev billiard table diagrams
- *[C.–Even-Zohar–Krishnan 2018]*,
  Crossing numbers of random 2-bridge Chebyshev knots
- *[C. 2021]*, *The Jones polynomials of 3-bridge knots via Chebyshev knots and billiard table diagrams*

- *[C. 2023]*,
  A lower bound on the average genus of a 2-bridge knot
- *[C.–Lowrance arXiv:2205.06122]*,
  The average genus of a 2-bridge knot is asymptotically linear
- *[C.–DiNardo–Lowrance–Raanes–Rivera–Steindl–Wanebo arXiv:2307.09399]*,
  The distribution of genera of 2-bridge knots